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UNIFIED MATHEMATICS.

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One of the first questions which a student asks at, or near, the beginning of his study of any branch of mathematics is, "What is it good for?" In most cases he is not satisfied unless he can be shown some specific utilitarian use to which his new knowledge may be put. With many students this attitude of mind persists throughout their study, which, for lack of a satisfying answer, is often closed at an earlier date than is desirable. Other students continue their mathematics solely for the use they can make of it as a tool in their study of some other science, as physics, for example.

Good pedagogy requires that the solution of concrete problems should form a large and important part of any course of mathematics, not merely because it will increase the student's interest in his work and at the same time give him the ability to use his mathematics as a tool, but because it will aid him in acquiring the power of close and accurate reasoning, of careful analysis, and of seeing the mathematical relationships of quantities — in a word — mathematical power. But with this end in view the mere results of the problems, which the utilitarian type of mind would prize so highly, become relatively insignificant, and the methods of attack and solution are of chief importance. In fact the extreme utilitarian, omitting everything which does not seem directly applicable to some "practical" problem, and in what he retains, focusing his attention upon the results, will defeat his own ends. For his students will be left with a few rules applicable in the particular cases studied, but they will have little or no mathematical ability — no power of analysis and no general theory for attacking new types of problems.

It must be acknowledged, however, that a student of good mathematical ability, or one who has been well taught and has followed a well arranged program of mathematical study, oftentimes is apparently unable to apply his mathematics to the study of a new science — in other words, he seems to lack that mathematical habit of mind at which we have aimed. In many cases, however, this failure is only apparent and the real difficulty is

with the method followed by the teacher of the new science. For example, we sometimes hear teachers of physics complain that their students cannot use mathematics in the solution of problems in physics. Before asking the students to make mathematical solutions of physical problems, the physicist must be sure that his students have been made familiar with the new concepts of physics, in their mathematical elements as well as their purely physical elements, and have seen that his teacher finds it of advantage to use mathematics in the solution of problems. In that way his own enthusiasm will be aroused to attempt the same methods, and he will not be called upon to apply his mathematics to data which are very dimly outlined in his mind. Indeed, one of the great difficulties in teaching mathematics is the selection of problems the concepts of which are thoroughly familiar to the student so that he can be asked to apply his mathematics intelligently. Yet the teacher of physics, sometimes forgetting how long it takes a student to become familiar with a new concept, even a very simple one, may override this fundamental principle of mathematical work—a clear knowledge of the data of a problem—and ask the student to apply his mathematics at too early a date. Not until he is sure that he has made his students appreciate the physical data has the teacher of physics any right to complain of their mathematical work.

While, then, the results of problems should not be regarded as of fundamental importance, the solution of problems, taken from all possible sources, must form a large part of any course in mathematics, since in this way the power of analysis is largely developed. This power of analysis—the recognition of the mathematical element in a problem—and a good understanding of the meaning of mathematical symbols and the laws governing their relations should be the two aims of a course in mathematics. Some students enjoy the problem work more than the theoretical work, and others care more for the theoretical work than the problem work, but both classes of students must study the subject on both its sides if they would not fall short of their possibilities, even in the particular part of the work in which they are the more interested.

In planning a course of mathematics, then, what order shall be used? In trying to answer this question we shall consider mathematics through a first course in calculus, *i. e.*, arithmetic, algebra, plane geometry, solid geometry, trigonometry, analytic geometry, differential calculus, and integral calculus. It is usual to teach these subjects separately and in the order in which they have been named, with the exception that the Algebra is often divided into two courses, the second course being placed after the course in plane geometry. This method divides this portion of the science up into distinct parts or units, and the student appreciates this division probably more strongly than he appreciates the unity of the whole science. How, otherwise, are we to explain the inability, or the unwillingness, of the average student to use the mathematics of one division in the work of an-

other division? For example, after a year's study of algebra, plane geometry is taken up. Many of the proofs in plane geometry may be expressed very neatly in algebraic form; also many propositions in geometry open up a field of problems for algebraic solution. Now it is well known that with many students it is most difficult, if not impossible, to make them use their algebra in this work. They seem to regard the algebra as if it were finished and laid aside, and not as a vital part of their mental training, to be used everywhere that it will be of advantage.

Recently, however, even in elementary algebra considerable use has been made of the graph in explaining results and in forecasting what may be expected as the answer to any problem. It may be noted in passing that, if the student is to derive the most benefit from this work, care should be taken to prevent him from regarding the graph as merely an interesting appendix at the end of the problem, but to make him think of it as an integral part of the solution. Here analytic geometry has aided algebra in the solution of its problems in a very valuable manner, and at the same time the barriers between these two branches of mathematics have been partially broken down, although the student, of course, does not appreciate this fact.

Again, when any single division of mathematics is developed by itself, such development is apt to be narrow and even to give a wrong impression. For example, analytic geometry may be studied without any use of the calculus, and a very compact and interesting body of methods and results may be put together. But the difficulty of dealing with any but the simplest curves before the methods of calculus are used has made analytic geometry and conic sections almost synonomous terms in the past. Moreover, the student's knowledge of the meaning of analytic geometry will in this way be defective, unless he goes on to the study of calculus, and he will be obliged throughout his life to use poorer tools than is necessary. For example, tangents and normals to curves are found by methods which are intrinsically those of calculus, but which can be applied only in the simplest cases, and even then with considerable superfluous labor, because of the lack of the notation of calculus. If it should be argued that the student would have two methods, *i. e.*, the method used in analytic geometry, and the method of calculus, the reply is the question: "Does any one, after he knows how to determine tangents and normals by calculus, ever resort to the longer and more laborious methods he used in analytic geometry?"

While the study of each part of mathematics by itself will give the student a good idea of the development of thought in that particular field — a point which may be regarded by some as of decided advantage — such classification of mathematics may be perfectly well postponed to a later date, when the student has a certain body of mathematics which he knows and wishes to classify. At this later date, if he is interested, the student may with considerable pleasure and profit study the historical development of each part of his subject, for he will then have the necessary mathematical

maturity, and having, moreover, knowledge of more than one field he can compare the advances in the various fields, and note how an advance in one field made possible an advance in another field.

Mathematics is a science, an old and fundamental science, which is being used more and more in the advancement of other sciences. But this increased use in other sciences can give us little aid in determining a method of teaching mathematics—in fact, a teacher keeping this view alone in mind might meet the same kind of failure that the mere utilitarian meets, and fail quite as completely to achieve his object. It would seem, then, as if we must study mathematics as a science in itself, but find some method of obviating difficulties such as have been noted above. In looking for a new method we will first note the attitude of the professional mathematician toward his own science. As he proceeds in the study of his subject he usually finds that some type of the work appeals to him especially and devotes the greater part, if not all, of his energy to that field. But this specialization would not be apt to take place till after the first course in calculus. Before that time, he may, to be sure, be more interested in the development of the theory than in its application to problems, or vice-versa. The main thing to be noticed, however, is that *he always tries to make his solution of any problem, theoretical or practical, as direct and simple as possible, and pays little or no attention to the particular field, or fields, from which he takes his method.* If, then, the mathematician does not permit himself to be restricted to any one field for his method, why should we teach mathematics in the old divisions, the boundary lines of which have proved difficult for so many students to cross? Why set up any boundary at all for the student? Teach him his mathemetics as a unit, the only way in which he will use it, and promise him at a later date to classify it as he wishes.

In co-operation with others the writer has recently had the opportunity of laying out and conducting such a unified course of mathematics for the first two and one-half years of an engineering school. The course takes the place of distinct courses which were formerly given in theory of eqations, plane and solid analytic geometry, differential calculus, integral calculus, and differential equations. The results with the classes have fully justified the experiment, and with each succeeding year the writer's belief in the desirability and the effectiveness of such courses has increased.

In arranging a unified course the order is *an order of difficulty in the problems,* and the theory is developed as rapidly as it is required for the solution of the problems. The power of analysis and the ability to do formal work can be as logically developed in one order as in another. In addition there is the decided advantage that the same method may occur several times with problems of increasing difficulty, and by this repetition a method, at first vague, may become clear and powerful. For example, the tangent to a curve is first found for a curve of the type

$$y=a_0x^n+a_1x^{n-1}+\ldots+a_{n-1}x+a_n,$$

where the right hand member is a rational algebraic polynomial of degree n, simple numerical cases being taken first; then for a curve of which the equation is algebraic, but more complex; and finally for curves of which the equations are transcendental. The first case requires only the formula for differentiating the algebraic polynomial; before the second case can be studied, the differentiation of any algebraic function must be studied; and before the last case can be studied, the differentiation of the transcendental functions must be considered. Here the whole subject of differentiation has been developed at the same time that a use for it has been shown, and the problem of the tangent has been studied at three different times with problems arranged in an ascending order of difficulty. Of course other problems would be carried on in this same manner with the development of the formulas of differentiation.

In a unified course there may be a saving of time, as no problem will be taken up till the best method of work has been developed for that problem. But such economy of time ought not to be regarded as the chief end to be attained by such a course. If it is so regarded, the course may be open to the criticism that some parts of mathematics, well and thoroughly studied before, may be slighted in the new course. As a matter of fact everything in the old courses, if desired, may be retained in the new course, and receive quite as much attention as in the old order, though one cannot but feel that some things as ordinarily given in the old courses could be omitted with no serious loss.

Again, the unified course, if too rapid advance is attempted, may be thought to make too great demands upon a student's power of assimilating new material. But it is in this very respect that the unified course is notably superior to the older style of course. In the latter course the particular new point is dwelt upon at such length in the effort to make the student understand it thoroughly before going on, as he will not take it up again, that he may find his work monotonous, and lose his interest. In the unified course, however, he takes up the new idea, studies it in an elementary way, and learns as much of it as possible without having the work become dull and uninteresting. He then goes on to some new ideas and after a time comes back to the study of the first idea with fresh pleasure in recognizing an old friend in new surroundings and with an increased appreciation of its full content. The old style course, by its intensive method, makes as great (if not greater) demands upon a student's power of assimilation, for it requires him to learn the new idea at once, while the unified course permits him to become acquainted with a new idea gradually, by meeting it at several different times, as illustrated by the example of the tangent line; and each time something is added to his previous knowledge, which is thus given time for a natural growth.

One other criticism of the unified course occurs to the writer at this time — the difficulty of a student referring to a text arranged for a unified course, when at some later time he wishes to use some formula or rule which he has forgotten. This difficulty, however, can easily be met by the compiling of a good index, and every student ought to learn the use of such an index, instead of shuffling the pages of his book till he finds what he wants. If the student of a unified course wishes to refer to some of the special texts of algebra, geometry, calculus, etc., it might be possible that he would not know what texts to consult — not know whether he ought to look in an analytic geometry or a calculus. But this difficulty seems so unlikely that it is hardly worth considering.

It is not to be assumed that all mathematics through the first course in calculus should be arranged in a single unified course. In fact, the break in going from the secondary school to the college seems to be an argument against such a large consolidation and to point to the desirability of at least two such unified courses.

Finally, such unified courses in the mathematics through the first course in Calculus are directly in line with what has long been done in the higher mathematics, as shown by the courses in analysis given by the French, and to some extent by the Germans, for advanced students. Surely a method which is so valuable there ought to be considered for other grades of work, unless the teaching of elementary mathematics is regarded as perfect at the present time; and the lively discussion of questions of teaching by the various associations of teachers makes one believe that the teachers are not so content with the present results.

JORDANUS NEMORARIUS AND JOHN OF HALIFAX.

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Cantor closes the first volume of his *Geschichte der Mathematik*[1] with the names of Leonard of Pisa and Jordanus Nemorarius, stating that these men mark a new period in the development of mathematical science. Cantor's discussion[2] of the part played by Jordanus in opening a new era rests largely on the Schonerus edition[3] of an anonymous mediaeval *Algorismus Demonstratus* which was long attributed to Regiomontanus, for no other reason apparently than that Regiomontanus made a copy of a Vienna manu-

1). Vol. I, 3d edition, p. 911.

2). Vol. II, 2d edition.

3). D. E. Smith, *Rara Arithmetica*, pp. 178-179. Curtze (note, p. 20, Einige Materialen zur Geschichte der Mathematischen Facultaet der alten Universitaet Bologna, S. Gherardi, translation by Max. Curtze, Berlin, (1871), states that there is a XIV cent. Ms. in the library of Basle.